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The proof of the upper bound in the multifractal formalism for chirps

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Abstract

The multifractal formalism for chirps is a formula conjectured by Jaffard. It describes the statistics of both the Hölder exponents H and the (chirp) oscillation exponents β characterizing the singular behavior involved in a given singular function. In that formula, the "chirp-type" Hölder spectrum $d(H,\beta)$ is related to oscillation spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$. For either $s \geqslant 0$ or $s \leqslant -m/p$, these spaces are a variation on the definition of Besov (or Sobolev) spaces. On the contrary the spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$ for $-m/p < s < 0$ cannot be sharply imbedded between Sobolev spaces, and thus are new spaces of really different nature. We prove that the multifractal formalism for chirps yields for any function an upper bound. Besides, this upper bound is optimal.

1. Introduction

Let us recall the definitions of Hölder regularity and Hölder spectrum. Let $F : \mathbb{R}^m \mapsto \mathbb{R}$ be a function.

Definition 1.

- $F \in C^h(x_0)$ for $h > 0$ if there exist a polynomial P of degree smaller than h and a constant C such that, in a neighborhood of x_0

$$|F(x) - P(x - x_0)| \leqslant C|x - x_0|^h. \tag{1}$$

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- The Hölder exponent $h_F(x_0)$ of F at x_0 is the supremum of all values of h such that $F \in C^h(x_0)$.
- The Hölder spectrum of F is the function $d(H)$ which associates to each positive H the Hausdorff dimension of the set $E^{(H)}$ of points x where $h_F(x) = H$ (by convention $dim(\emptyset) = -\infty$).
- $F \in C^h(\mathbb{R}^m)$ for $h > 0$ if (1) holds for any x and x_0 in $\mathbb{R}^m$ with a uniform constant C.
- The global Hölder regularity of F is the supremum of all values of h such that $F \in C^h(\mathbb{R}^m)$.

The (standard) multifractal formalism was introduced by Frisch and Parisi [10] in the context of fully developed turbulence. It proposes to compute the Hölder spectrum of a function F using the formula

$$d(H) = \inf_{p>0}(pH - \eta(p) + m), \tag{2}$$

where $\eta(p)$ is defined by $\int_{\mathbb{R}^m} |F(x+l) - F(x)|^p \, dx \sim |l|^{\eta(p)}$ when $l \mapsto 0$. Note that the study of this Legendre transform was already advocated in the seminal paper of Mandelbrot [18]. An alternative formula based on the local maxima of the wavelet transform was proposed by Arneodo, Bacry, and Muzy (see [1]). Jaffard [12], proved that formula (2) and the wavelet-based formula yield the same exponent $\eta(p)$ which can be deduced from the functional Besov-type spaces to which the function F belongs, i.e.,

$$\eta(p) = \sup\{s\colon\ F \in B_p^{s/p,\infty}(\mathbb{R}^m)\}. \tag{3}$$

The definition of a Besov space will be recalled in (9).

The validity of (2) has been proved for a large class of selfsimilar functions (see [2,4–9,12]). Jaffard [14], proved that (2) holds for quasi-all functions, i.e., outside a set of the first class of Baire.

In [16], a counter-example shows that the (standard) multifractal formalism fails in the case where the function displays very oscillating behaviors. The Hölder exponent is not precise enough, in the sense that it does not take into account the local oscillations of the function. Indeed a given Hölder exponent H at x_0 allows for different behaviors near x_0: for instance cusp-like singularities, such as $|x - x_0|^H$ or very oscillatory behaviors, such as

$$F_{H,\beta}(x) = |x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right) \tag{4}$$

for $\beta > 0$. The functions $F_{H,\beta}$ are the most simple examples of chirps at x_0. In signal analysis, this notion is expected to give a model for functions whose "instantaneous frequency" increases fast at some time (see [17]).

A careful study of the (standard) multifractal formalism shows that this formalism is only adapted to "cusp-type" singularities (i.e., singularities with oscillation exponent $\beta = 0$). Contrary to functions with cusp singularities, the primitive of the oscillating function (4) has an Hölder exponent $H + 1 + \beta$ at x_0 which is different from $H+1$ (the gain of regularity is not 1 as might be expected but $\beta+1$). This remark motivated the following definition introduced by Meyer [17].

Definition 2. Let $h\geqslant 0$ and $\beta>0$. A function F in $L^\infty(\mathbb{R}^m)$ is a (h,β)-type chirp at x_0 if $\forall n\in\mathbb{N}$, F can be written as a finite sum of partial derivatives of order n of functions which belong to $C^{h+n(1+\beta)}(x_0)$.

The interior of the set of points (h,β) such that a function F is a (h,β)-type chirp at x_0 is always a domain of the form $h<h_F(x_0)$, $\beta<\beta(x_0)$ (see [16]). The non-negative real number $\beta(x_0)$ is called the chirp exponent at x_0, i.e.,

$$\beta(x_0)=\sup\{\beta;\exists h<h_F(x_0) \text{ such that } F \text{ is a } (h,\beta)\text{-type chirp at } x_0\}. \tag{5}$$

If we want to study chirps located in a signal, we are naturally led to define a spectrum of "chirp-type" Hölder singularities as follows.

Definition 3. The spectrum $d(H,\beta)$ of chirp-type Hölder singularities of a function F is the Hausdorff dimension of the set $E^{(H,\beta)}$ of points where F has chirp exponents (H,β).

Jaffard conjectured by thermodynamic arguments a new multifractal formalism adapted to chirps. He proposed the use of oscillation spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$ (see Definition 4) in order to capture the oscillating behaviors which are left undetected by Sobolev or Besov spaces.

Let

$$\zeta(p,s')=\sup\{s;F\in\mathcal{O}_p^{s/p,s'/p}(\mathbb{R}^m)\}. \tag{6}$$

The formula of Jaffard asserts that

$$d(H,\beta)=\inf_{s',p}(Hp-(1+\beta)s'-\zeta(p,s')). \tag{7}$$

It is called the multifractal formalism for chirps (or for chirp-type Hölder singularities). Jaffard checked its validity for lacunary wavelet series (for which the standard multifractal formalism was wrong). Arneodo et al. [3] constructed a family of wavelet series for which (7) holds.

In the next section, we recall both the wavelet characterization of Besov spaces, oscillation spaces and chirps. We also give the heuristic argument from which formula (7) was derived.

In the third section, we recall the embeddings between oscillation and Besov spaces.

In the fourth section, we will demonstrate the upper bound for the new formalism.

2. The heuristic derivation of the multifractal formalism for chirps

Suppose that the function $F:\mathbb{R}^m\rightarrow\mathbb{R}$ is known by the explicit knowledge of its coefficients in a given basis (see [20]). Let $\psi^{(i)}$, $i=1,\dots,2^m-1$, be a family of 2^m-1 smooth wavelets such that the $2^{mj/2}\psi^{(i)}(2^jx-k)$, $i=1,\dots,2^m-1$, $j\in\mathbb{Z}$, $k\in\mathbb{Z}^m$, form an orthonormal basis of $L^2(\mathbb{R}^m)$. We will use a L^∞ normalization for wavelets, so that we write

$$F(x)=\sum_{i,j,k} C_{j,k}^{(i)}\psi^{(i)}(2^jx-k), \tag{8}$$

where $C_{j,k}^{(i)}=2^{mj}\int_{\mathbf{R}^m}F(x)\psi^{(i)}(2^jx-k)\,dx$. We will from now on use the following simpler notations; λ and λ' will denote, respectively, the cubes $\lambda_{j,k}=k2^{-j}+[0,2^{-j}]^m$ and $\lambda_{j',k'}=k'2^{-j'}+[0,2^{-j'}]^m$, C_λ will denote the coefficient $C_{j,k}^{(i)}$, and ψ_λ will denote the wavelet $\psi^{(i)}(2^jx-k)$ (note that we "forget" to write the index i of the wavelet, which is of no consequence). Recall that F belongs to the Besov space $B_p^{s,q}(\mathbb{R}^m)$ with $p>0$ and $q>0$ if (see [20])

$$2^{sj}2^{-mj/p}\left(\sum_k |C_\lambda|^p\right)^{1/p}:=\varepsilon_j \quad \text{with } \varepsilon_j\in l^q. \tag{9}$$

The spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$ are function spaces that have been introduced by Jaffard [13] in order to quantify with one parameter (or a few) the degree of correlations between positions of large wavelet coefficients.

Definition 4. Let $p>0$, and $s,s'\in\mathbb{R}$.

A function F belongs to the oscillation space $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$ if its wavelet coefficients satisfy

$$\exists C>0\quad \forall j\geqslant 0\quad 2^{sj}\left(\sum_k \sup_{\lambda'\subset\lambda}|C_{\lambda'}2^{s'j'}|^p\right)^{1/p}\leqslant C. \tag{10}$$

Note that, if $p=\infty$, this condition becomes

$$\exists C>0\quad \forall j\geqslant 0\quad 2^{sj}\sup_{j'\geqslant j}|C_{\lambda'}2^{s'j'}|\leqslant C.$$

The left-hand side defines the $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$ semi-norm.

Jaffard and Meyer [17, Theorem 4.2], characterized chirps in terms of estimates on the size of the wavelet coefficients.

Proposition 1. *F is a (h,β)-type chirp at x_0 with global Hölder regularity $r>0$ if and only if for any $N\geqslant 0$*

- *if $|k2^{-j}-x_0|\leqslant 2^{-j}$ then $|C_\lambda|\leqslant C_N 2^{-Nj}$*
- *if $|k2^{-j}-x_0|^{1+\beta}\leqslant 2^{-j}\leqslant|k2^{-j}-x_0|$ then $|C_\lambda|\leqslant C_N|k2^{-j}-x_0|^h\left(\frac{|k2^{-j}-x_0|^{1+\beta}}{2^{-j}}\right)^N$*
- *if $|k2^{-j}-x_0|^{1+\beta}\geqslant 2^{-j}$ then $|C_\lambda|\leqslant C|k2^{-j}-x_0|^h\left(\frac{2^{-j}}{|k2^{-j}-x_0|^{1+\beta}}\right)^r$.*

It follows that if F is a (h,β)-type chirp at x_0, its wavelet coefficients are of the order of magnitude of $|k2^{-j}-x_0|^h$ near the curve $|k2^{-j}-x_0|^{1+\beta}\sim 2^{-j}$ and decay fast away from this curve.

Let us recall the heuristic argument from which formula (7) was derived. Though this argument cannot be transformed into a correct mathematical proof, it shows at least why that formula can be expected to hold. We estimate for each (H,β) the contribution of the chirps of exponents (H,β) to the quantity

$$\sum_k \sup_{\lambda'\subset\lambda} |C_{\lambda'}|^p 2^{s'j'}. \tag{11}$$

Consider a cube λ of size 2^{-j} which contains a chirp of exponents (H,β) at x_0. Then the coefficients $|C_{\lambda'}|$ are of the order of magnitude of $|k'2^{-j'}-x_0|^H$ near the curve $|k'2^{-j'}-x_0|^{1+\beta}\sim 2^{-j'}$ and decay fast away from this curve. Let $\lambda'\subset\lambda$. Since $x_0\in\lambda$ then $|k'2^{-j'}-x_0|\leqslant 2^{-j}$, so $|k'2^{-j'}-x_0|^{1+\beta}\leqslant 2^{-j(1+\beta)}$. It follows that the wavelet coefficients $C_{\lambda'}$ for $\lambda'\subset\lambda$ are negligible as long as $2^{-j'}>2^{-j(1+\beta)}$, i.e., as long as $j'<j(1+\beta)$. When $j'\sim j(1+\beta)$, for some values of k',

$$|C_{\lambda'}|\sim|k'2^{-j'}-x_0|^H \quad\text{and}\quad |k'2^{-j'}-x_0|^{1+\beta}\sim 2^{-j'}.$$

So

$$|C_{\lambda'}|\sim 2^{-j'\frac{H}{1+\beta}}\sim(2^{-j})^H.$$

So that

$$\sup_{\lambda'\subset\lambda}|C_{\lambda'}|^p 2^{s'j'}\sim 2^{-j'\left(\frac{Hp}{1+\beta}-s'\right)}\sim 2^{-j(Hp-(1+\beta)s')}$$

(as long as $s'\leqslant pH/(1+\beta)$, else the supremum is infinite). The contribution of the chirps of exponents (H,β) to quantity (11) is thus

$$2^{jd(H,\beta)}2^{-j(Hp-(1+\beta)s')}=2^{-j(Hp-(1+\beta)s'-d(H,\beta))},$$

where $d(H,\beta)$ was given in Definition 3. When $j\mapsto\infty$, the main contribution is obtained by the couple (H,β) realizing the infimum of $Hp-(1+\beta)s'-d(H,\beta)$; hence the heuristic formula

$$\zeta(p,s')=\inf_{H,\beta}\,(Hp-(1+\beta)s'-d(H,\beta)). \tag{12}$$

This formula is not the one we are looking for since we know $\zeta(p, s')$ and we look for $d(H, \beta)$; but if (12) holds and $d(H, \beta)$ is convex, $d(H, \beta)$ is recovered by an inverse Legendre transform formula which yields (7).

3. Embeddings between oscillation and Besov spaces

Jaffard [13,15] proved that the spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$ for either $s \geqslant 0$ or $s \leqslant -m/p$ are a variation on the definition of Besov (or Sobolev) spaces. On the contrary the spaces $\mathcal{O}_p^{s,s'}(\mathbb{R}^m)$ for $-m/p < s < 0$ cannot be sharply imbedded between Sobolev spaces, and thus are new spaces of really different nature. Using the convention $C^{s'}(\mathbb{R}^m) = B_\infty^{s',\infty}(\mathbb{R}^m)$ even when s' is an integer, i.e., $F \in C^{s'}(\mathbb{R}^m)$ if and only if its wavelet coefficients satisfy the condition

$$|C_\lambda| \leqslant C 2^{-s'j}, \tag{13}$$

Jaffard proved the following results.

Proposition 2.

$$\forall \varepsilon \geqslant 0, \quad \forall \varepsilon' \geqslant 0, \quad \mathcal{O}_p^{s+\varepsilon, s'+\varepsilon'}(\mathbb{R}^m) \subset \mathcal{O}_p^{s,s'}(\mathbb{R}^m), \tag{14}$$

$$\forall \varepsilon > 0, \quad \mathcal{O}_p^{s-\varepsilon, s'+\varepsilon}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{s,s'}(\mathbb{R}^m). \tag{15}$$

1. *If* $s > 0$, *then* $\mathcal{O}_p^{s,s'}(\mathbb{R}^m) = B_p^{s+s'+m/p,\infty}(\mathbb{R}^m)$.
2. $B_p^{s'+m/p,p}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{0,s'}(\mathbb{R}^m) \hookrightarrow B_p^{s'+m/p,\infty}(\mathbb{R}^m)$.
3. *If* $-m/p < s < 0$, *then*

$$B_p^{s'+m/p,p}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{s,s'}(\mathbb{R}^m) \hookrightarrow C^{s'}(\mathbb{R}^m)$$

   and

$$C^{s+s'+m/p}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{s,s'}(\mathbb{R}^m) \hookrightarrow B_p^{s+s'+m/p,\infty}(\mathbb{R}^m)$$

4. *If* $s \leqslant -m/p$ *then* $\mathcal{O}_p^{s,s'}(\mathbb{R}^m) = C^{s'}(\mathbb{R}^m)$.

Furthermore, in Cases 2 *and* 3, *the embeddings are optimal.*

Such embeddings were given for a definition of the oscillation spaces which differs slightly from the one we choose here. So we prefer here to rewrite the proof of these embeddings. However, we refer to [15] for the proof of the optimality of the embeddings.

The first point of Proposition 2 is straightforward. The second point follows from the fact that if $j' \geqslant j$ then for any positive ε, $2^{(j'-j)\varepsilon} \geqslant 1$ so that

$$\forall j \quad 2^{sjp} \sum_k \sup_{\lambda' \subset \lambda} |C_{\lambda'}|^p 2^{s'j'p} \leqslant 2^{(s-\varepsilon)jp} \sum_k \sup_{\lambda' \subset \lambda} |C_{\lambda'}|^p 2^{(s'+\varepsilon)j'p}.$$

Now, we prove the embeddings of oscillation spaces into Besov spaces. Since $|C_\lambda| 2^{js'} \leqslant \sup_{\lambda' \subset \lambda} |C_{\lambda'}| 2^{j's'}$ then $|C_\lambda| \leqslant \sup_{\lambda' \subset \lambda} |C_{\lambda'}| 2^{(j'-j)s'}$. It follows that

$$\forall j \quad 2^{(s+s'+m/p)j} 2^{-mj/p} \left(\sum_k |C_\lambda|^p \right)^{1/p} \leqslant 2^{sj} \left(\sum_k \sup_{\lambda' \subset \lambda} |C_{\lambda'}|^p 2^{j's'p} \right)^{1/p},$$

so that $\forall s, s', p,\ \mathcal{O}_p^{s,s'}(\mathbb{R}^m) \hookrightarrow B_p^{s+s'+m/p,\infty}(\mathbb{R}^m)$. Furthermore, if (10) holds, applying this bound for $j = 0$ yields $\sup_{\lambda'} |C_{\lambda'} 2^{s'j'}| \leqslant C$, so that

$$\forall s, s', p, \quad \mathcal{O}_p^{s,s'}(\mathbb{R}^m) \hookrightarrow C^{s'}(\mathbb{R}^m). \tag{16}$$

In order to prove the converse embeddings, we remark that we can bound $\sup_{\lambda' \subset \lambda} (|C_{\lambda'}| 2^{j's'})^p$ either by

$$\sum_{j' \geqslant j} \sum_{\lambda' \subset \lambda} (|C_{\lambda'}| 2^{j's'})^p \tag{17}$$

or by

$$\sup_{j' \geqslant j} \sup_{\lambda'} (|C_{\lambda'}| 2^{j's'})^p \tag{18}$$

(where $\sup_{\lambda'}$ is taken on all dyadic cubes of size $2^{-j'}$).

If we use (17) as upper bound, we obtain

$$2^{spj} \sum_k \sup_{\lambda' \subset \lambda} |C_{\lambda'}|^p 2^{j's'p} \leqslant 2^{spj} \sum_k \sum_{j' \geqslant j} \sum_{\lambda' \subset \lambda} |C_{\lambda'}|^p 2^{j's'p}.$$

The latest term is equal to

$$2^{spj} \sum_{j' \geqslant j} \sum_{k'} |C_{\lambda'}|^p 2^{j's'p}. \tag{19}$$

- If $s = 0$, then (19) is equal to $\sum_{j' \geqslant j} \sum_{k'} |C_{\lambda'}|^p 2^{j's'p}$. Applying this bound for $j = 0$ yields

$$B_p^{s'+m/p,p}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{0,s'}(\mathbb{R}^m). \tag{20}$$

- If F belongs to $B_p^{s+s'+m/p,\infty}(\mathbb{R}^m)$, $2^{(s+s')pj'} \sum_{k'} |C_{\lambda'}|^p \leqslant C$ so that, if $s > 0$, (19) is bounded. Thus if $s > 0$, $B_p^{s+s'+m/p,\infty}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{s,s'}(\mathbb{R}^m)$.

- If $s<0$, using (20) and (14), it follows that
$$B_p^{s'+m/p,p}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{0,s'}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{s,s'}(\mathbb{R}^m).$$

Now if we bound $\sup_{\lambda' \subset \lambda} (|C_{\lambda'}|2^{j's'})^p$ by (18), since (18) does not depend on k,

$$2^{spj} \sum_k \sum_{\lambda' \subset \lambda} |C_{\lambda'}|^p 2^{j's'p} \leqslant 2^{mj} 2^{spj} \sup_{j' \geqslant j} \sup_{\lambda'} |C_{\lambda'}|^p 2^{j's'p}. \tag{21}$$

- If $s \leqslant -m/p$ and F belongs to $C^{s'}(\mathbb{R}^m)$, $\sup_{\lambda'} |C_{\lambda'}|^p 2^{j's'p} \leqslant C$ so that the right-hand side of (21) is bounded by $C2^{mj}2^{spj} \leqslant C$ and $C^{s'}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{s,s'}(\mathbb{R}^m)$.
- If $s > -m/p$ and $F \in C^{s+s'+m/p}(\mathbb{R}^m)$, $|C_{\lambda'}| \leqslant C2^{-(s+s'+m/p)j'}$, so that the right-hand side of (21) is bounded by C and $C^{s+s'+m/p}(\mathbb{R}^m) \hookrightarrow \mathcal{O}_p^{s,s'}(\mathbb{R}^m)$.

4. The proof of the upper bound for the multifractal formalism for chirps

Jaffard and Meyer [17, Theorem 4.2], characterized chirps using two-microlocal spaces; Recall (see [11]) that a distribution F belongs to the two-microlocal space $C_{x_0}^{t,t'}$ if there exists $C>0$ such that the wavelet coefficients of F satisfy, for $|x_0 - k2^{-j}|$ close enough to 0,

$$|C_\lambda| \leqslant C2^{-j(t+t')}(2^{-j} + |x_0 - k2^{-j}|)^{-t'}, \tag{22}$$

i.e.,

$$|C_\lambda| \leqslant C2^{-jt}(1 + |2^j x_0 - k|)^{-t'}. \tag{23}$$

Proposition 3. *F is a (h, β)-type chirp at x_0 with global Hölder regularity $r>0$ if and only if F belongs to all the two-microlocal spaces $C_{x_0}^{t,t'}$ for $t+t' \leqslant r$ and $(\beta+1)t + \beta t' \leqslant h$.*

Jaffard and Meyer also characterized two-microlocal spaces in terms of local Hölder type conditions (see [17, Theorem 1.2]). Let B_ρ be the ball $|x - x_0| \leqslant \rho$ and Γ_ρ the annulus $\rho \leqslant |x - x_0| \leqslant 3\rho$. Let A be a set and $s \in \mathbb{R}$. By definition, a function f belongs to $C^s(A)$ if it is the restriction to A of a function F in $C^s(\mathbb{R}^m)$. The norm of f is then the infimum of all possible norms of F in $C^s(\mathbb{R}^m)$.

Proposition 4. *If t' is negative, a distribution f defined in a neighborhood of x_0 belongs to $C_{x_0}^{t,t'}$ if and only if there exists $C>0$ such that*

$$\|f\|_{C^{t+t'}(B_\rho)} \leqslant C\rho^{-t'}. \tag{24}$$

If t' is positive, f belongs to $C^{t,t'}_{x_0}$ if and only if there exists $C>0$ such that

$$||f||_{C^{t+t'}(\Gamma_\rho)} \leqslant C\rho^{-t'} \tag{25}$$

and

$$f \in C^t(\mathbb{R}^m). \tag{26}$$

In view of (16), in order to prove the upper bound in the multifractal formalism for chirps, it suffices to show the following result.

Theorem 1. *If $F \in \mathcal{O}^{s,s'}_p(\mathbb{R}^m)$ with s' smaller than the global Hölder regularity r of F then*

$$d(H,\beta) \leqslant p(H-(1+\beta)s'-s).$$

Proof. Since $F \in \mathcal{O}^{s,s'}_p(\mathbb{R}^m)$ then there exists $C>0$ such that

$$\forall j, \quad 2^{sj}\left(\sum_k \sup_{\lambda' \subset \lambda} |C_{\lambda'} 2^{s'j'}|^p\right)^{1/p} \leqslant C. \tag{27}$$

Let d be such that $0<d\leqslant m$ and $B^d_{j,k}$ be the ball centered on $k2^{-j}$ and of size

$$diam(B^d_{j,k}) = j^{-2/d} 2^{sjp/d} \sup_{\lambda' \subset \lambda} |C_{\lambda'} 2^{s'j'}|^{p/d}.$$

Then (27) can be rewritten as

$$\forall j, \quad \sum_k (diam(B^d_{j,k}))^d \leqslant \frac{C}{j^2}. \tag{28}$$

Let $A^d_j = \bigcup_k B^d_{j,k}$. Relation (28) implies that the d-Hausdorff measure of $A^d := \limsup A^d_j$ is 0.

Proposition 5. *If $x_0 \notin A^d$ then F belongs to all the two-microlocal spaces $C^{t,t'}_{x_0}$ for $t+t' \leqslant s'$ and $t<s+s'+d/p$.*

Proof. Let $x_0 \notin A^d$, then there exists j_0 such that $\forall j \geqslant j_0$, $\forall k$, $x_0 \notin B^d_{j,k}$ so that

$$\forall j \geqslant j_0, \quad \forall k, \forall \lambda' \subset \lambda\colon\ |C_{\lambda'}| \leqslant 2^{-s'j'} 2^{-js} |x_0 - k2^{-j}|^{d/p} j^{2/p}. \tag{29}$$

If furthermore $|x_0 - k2^{-j}| \leqslant C2^{-j}$, then

$$\forall \lambda' \subset \lambda\colon\ |C_{\lambda'}| \leqslant C 2^{-s'j'} 2^{-j(s+d/p)} j^{2/p}. \tag{30}$$

Hence, if $|x_0 - k2^{-j}| \leqslant C2^{-j}$, then

$$\forall \lambda' \subset \lambda\colon\ |C_{\lambda'}| \leqslant C 2^{-s'j'} 2^{-j(s+d/p-\varepsilon)} \quad \forall \varepsilon > 0. \tag{31}$$

Let now $0<\rho<1/8$, there exists a unique $j\in\mathbb{N}^*$ such that $\frac{1}{8}2^{-j}\leqslant\rho<\frac{1}{4}2^{-j}$ so that $B_\rho\subset[|x-x_0|\leqslant\frac{1}{4}2^{-j}]$ and $\Gamma_\rho\subset[\frac{1}{8}2^{-j}\leqslant|x-x_0|\leqslant\frac{3}{4}2^{-j}]$. There exist at most two cubes of the form λ that cover B_ρ and Γ_ρ. Thus, if there exists $C>0$ such that for any j large enough and for any k such that $|x_0-k2^{-j}|\leqslant C2^{-j}$, the condition

$$||f||_{C^{t+t'}(\lambda)}\leqslant C(2^{-j})^{-t'} \tag{32}$$

holds, then relations (24) and (25) hold too. Remark that (32) is equivalent to

$$\forall\lambda'\subset\lambda:\ |C_{\lambda'}|\leqslant C2^{-(t+t')j'}2^{jt'}. \tag{33}$$

Remark also that $2^{j'a}2^{jb}\leqslant 2^{j'A}2^{jB}$ for all j large enough and $j'\geqslant j$ if and only if $a\leqslant A$ and $a+b\leqslant A+B$. It follows from (31) that relations (24) and (25) hold for any t and t' satisfying $t+t'\leqslant s'$ and $t<s+s'+d/p$. To achieve the proof of Proposition 5, we should verify (26) when t' is positive. From the convexity of the two-microlocal domain of F at x_0, i.e., $\{(t,t');\ F\in C^{t,t'}_{x_0}\}$, and the embeddings

$$\forall\delta>0,\quad C^{t,t'}_{x_0}\subset C^{t-\delta,t'+\delta}_{x_0} \tag{34}$$

and

$$\forall\delta>0,\quad C^{t,t'}_{x_0}\subset C^{t,t'-\delta}_{x_0} \tag{35}$$

it follows that F belongs to all the two-microlocal spaces $C^{t,t'}_{x_0}$ for $t+t'\leqslant s'$ and $t<s+s'+d/p$ if and only if F belongs to all the two-microlocal spaces $C^{s+s'+d/p-\varepsilon,-s-d/p+\varepsilon}_{x_0}$ for any ε close enough to 0. So, if $t':=-s-d/p+\varepsilon$ is positive then $t:=s+s'+d/p-\varepsilon=s'-t'\leqslant s'$. So in view of (16), we obtain $F\in C^t(\mathbb{R}^m)$.

Let us now achieve the proof of Theorem 1. Let $\varepsilon'>0$, we will show that if $x_0\notin A^d$ and

$$d=p(H-(1+\beta)s'-s) \tag{36}$$

then $x_0\notin E^{(H-\varepsilon',\beta)}$. Let $\varepsilon>0$ and $(T,T')=(s+s'+d/p-\varepsilon/2,-s-d/p)$. Since $T<s+s'+d/p$ and $T+T'\leqslant s'$ then $F\in C^{T,T'}_{x_0}$. Remark now that since $s'\leqslant r$ then $T+T'\leqslant r$. Nevertheless, (36) implies that $(\beta+1)T+\beta T'=H-(\beta+1)\varepsilon/2$ which is larger than $H-\varepsilon'$ for $\varepsilon<2\varepsilon'/(\beta+1)$. Hence $x_0\notin E^{(H-\varepsilon',\beta)}$. As a consequence, we have

$$\forall\varepsilon'>0,\quad E^{(H-\varepsilon',\beta)}\subset A^d.$$

Thus

$$\forall\varepsilon'>0,\quad d(H-\varepsilon',\beta)\leqslant p(H-(1+\beta)s'-s).$$

Therefore

$$\forall\varepsilon'>0,\quad d(H,\beta)\leqslant p((H+\varepsilon')-(1+\beta)s'-s).$$

This yields Theorem 1.

The upper bound in (7) is optimal since it becomes an equality in the case of lacunary wavelet series (see [13]). Besides Arneodo et al. [3] constructed an other family of wavelet series for which (7) holds.

Remark. There is a slightly different definition of oscillation exponent β which agrees with the definition of the chirp oscillation exponent for functions such as (4), and which is stable under the addition of a 'smooth noise'. Let $h_t(x_0)$ denote the Hölder exponent of the fractional primitive of order t at x_0 of a function F. More precisely, if F is locally bounded, we denote by $h_t(x_0)$ the Hölder exponent of the function $F_t := (Id - \Delta)^{-t/2}(\phi F)$ where ϕ is a C^∞ compactly supported function satisfying $\phi(x_0) = 1$. In the case of the function

$$F_o := |x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right) + O(|x - x_0|^{H'}),$$

where $H' > H$, for t small enough, $h_t(x_0) = H + (1 + \beta)t$. The increase of the pointwise Hölder regularity at x_0 after a fractional integration of very small order t is $(1 + \beta)t$. This remark motivated the following definition of Arneodo et al. [3].

Definition 5. Let $F : \mathbb{R}^m \to \mathbb{R}$ be a bounded function. The oscillating singularity exponents of F at a point x_0 are defined by

$$(H, \beta_o) = \left(h_F(x_0), \frac{\partial}{\partial t} h_t(x_0)\bigg|_{t=0} - 1\right). \tag{37}$$

These exponents belong to $[0, +\infty] \times [0, +\infty]$.

In [19], Melot obtained an optimal upper bound for $d(H, \beta_o)$ for functions in Besov spaces.

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